Risk-Neutral Modeling of Emission Allowance Prices and Option Valuation

René Carmona [*]   Juri Hinz [†]

Abstract

The existence of mandatory emission trading schemes in Europe and the US, and the increased liquidity of trading on futures contracts on CO_2 emissions allowances, led naturally to the next step in the development of these markets: these futures contracts are now used as underliers for a vibrant derivative market. In this paper, we give a rigorous analysis of a simple risk-neutral reduced-form model for allowance futures prices, demonstrate its calibration to historical data, and show how to price European call options written on futures contracts.

Key words Emission derivatives, Emissions markets, Cap-and-trade schemes, Environmental Finance.

1 Introduction

Global warming and environmental problems continue to challenge policy makers. In part because of the success of the U.S. Acid Rain Program, cap-and-trade systems are now considered as one of the most promising ways to combat climate change on an international scale. The core principle of such a mechanism is based on an authority which allocates fully tradable credits among emission sources and sets a penalty being paid per unit of pollutant which is not offset by a credit at the end of a pre-determined period. The idea is that the introduction of emission trading leads to price discovery of permits which helps to identify and to exercise the cheapest emission abatement measures. For this reason, market-based mechanisms for emission reduction are supposed to yield pollution control in the cheapest way for the society. Notwithstanding the fact that the rigorous equilibrium analysis from [4] and [5] confirm that social optimality does not necessarily mean that the scheme is cheap for consumers, emission trading should be considered as a cost-efficient and effective tool.

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[*] Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544. Also with the Bendheim Center for Finance and the Applied and Computational Mathematics Program. (*rcarmona@princeton.edu*). Partially supported by NSF-FRG-DMS-0455982 and DMS-0806591

[†] National University of Singapore, Department of Mathematics, 2 Science Drive, 117543 Singapore (*mathj@nus.edu.sg*), Partially supported by WBS R-703-000-020-720 / C703000 of the Risk Management Institute and by WBS R-146-000-107-133 of the National University of Singapore

By its very nature, the regulatory framework of any mandatory cap-and-trade system involves its participants in a risky business, necessarily creating the need for appropriate risk management. Trading of certificates from a mandatory scheme is typically accompanied by an active secondary market where diverse emission-related financial assets (futures) including a fast-growing variety of their derivatives are traded. In this work, we address the problem of risk neutral modeling of emission certificates. Our approach can be considered as a reduced form modeling intended to describe evolution of emission-related assets from the perspective of risk-neutral dynamics. With this contribution, we suggest a quantitative framework suitable for pricing diverse emission-linked derivative instruments.

Despite the large number of pieces in the popular press and numerous speculative articles in magazines, the scientific literature on cap-and-trade systems is rather limited, especially if one restricts ourselves to quantitative analysis. For the sake of completeness, we briefly review the publications related to our contribution. The *economic theory* of allowance trading can be traced back to [9] and [14] whose authors proposed a market model for the public good *environment* described by tradable permits. *Dynamic allowance trading* is addressed in [8], [22], [16], [12], [17], [20], [13] and in the literature cited therein. *Empirical evidence from existing markets* is discussed in [**?**]. This paper suggests economic implications and hints at several ways to model spot and futures allowance prices, whose detailed interrelations are investigated in [23] and [24]. There, the demand for derivative instruments in emission markets is also addressed. In [1] characteristic properties for financial time series are observed for prices of emission allowances from the mandatory European Scheme EU ETS. Furthermore, a Markov switch and AR-GARCH models is suggested. The work [15] considers also tail behavior and the heteroscedastic dynamics in the returns of emissions allowance prices. *Dynamic price equilibrium and optimal market design* are investigated in [4] which provides a mathematical analysis of the market equilibrium and uses optimal stochastic control to show social optimality. Based on this approach, [5] discusses price formation for goods whose production is affected by emission regulations. In this setting, an equilibrium analysis confirms the existence of the so-called windfall profits (see [19]) and provides quantitative tools to analyze alternative market designs, which are applied in [3] to optimize a cap-and-trade mechanism for a proposed Japanese emission trading scheme. *Pricing of options* was addressed only recently. The paper [7] discusses an endogenous emission permit price dynamics within equilibrium setting and elaborates on valuation of European option on emission allowances. The work [18] and the dissertation [25] deal with the the risk-neutral allowance price formation within EU ETS. Here, utilizing equilibrium properties, the price evolution is treated in terms of marginal abatement costs and optimal stochastic control. Also the work [6] is devoted to option pricing within EU ETS. The authors suppose that the drift of the underlying is related to a hidden variable which describes the overall market position in allowance contracts and make use of filtering techniques to derive option price formulas which reflect specific allowance banking regulations, valid in the EU ETS.

The present paper is organized as follows. First, we discuss a model of an emission market within one compliance period. Section 2 introduces a class of risk neutral models based on the analysis of diffusion martingales ending up with two values. Section 3 investigates historical model calibration for one of these models. In the second part of the paper, we generalize our analysis to a more realistic multi-period framework which incorporates generic features of the real-world emission markets. Section 4 provides the modifications necessary to extend the results of the one-period model and we show how to price European options on allowance futures prices in this general set-up.

2 Risk Neutral Model for a Single Compliance Period

In this section, we consider a simple model for an abstract emission market. We restrict ourselves to a single compliance period, say $[0, T]$. The more realistic case of a multi-period models is treated in Section 4.

In the one-period setting, credits are allocated at the beginning of the period in order to enable allowance trading until time T and to encourage agents to exercise efficient abatement strategies. At the compliance date T, market participants cover their emissions by redeeming allowances, or pay a penalty π per unit of pollution not offset by credits. In this one-period model, unused allowances expire worthless as we do not allow for banking into the next period. Under natural assumptions, equilibrium analysis shows that the allowance price A_T at compliance date T is a random variable taking only the values 0 and π (see [4] and [5]). More precisely, if the market remains under the target pollution level, then the price approaches 0. Otherwise, the allowance price tends to the penalty level π.

All the relevant asset price evolutions are assumed to be given by adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, on which we fix an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ which we call the spot martingale measure.We denote by $(A_t)_{t\in[0,T]}$ the price process of a future contract with maturity date T written on the allowance price. Given the digital nature of the terminal allowance price A_T, the central object of our study is the event $N \subset \mathcal{F}_T$ of non-compliance which settles the $\{0, \pi\}$-dichotomy of the terminal futures price by

$$A_T = \pi 1_N. \tag{1}$$

Furthermore, a standard no-arbitrage argument shows that the futures price $(A_t)_{t\in[0,T]}$ needs to be a martingale for the spot martingale measure $\mathbb{Q}$. Hence, the problem of allowance price modeling reduces to the appropriate choice of the martingale

$$A_t = \pi \mathbb{E}^{\mathbb{Q}}(1_N \,|\mathcal{F}_t), \quad t \in [0, T].$$

There are many candidates for such a process, but no obvious choice seems to be versatile enough for the practical requirements described below. An important requirement is the need to match the observed volatility structure. For a practitioner trying to calibrate at time $\tau \in [0, T]$ a model for the martingale $(A_s)_{s\in[\tau,T]}$ which finishes at 0 or π, the minimum requirements are to match the price observed at time τ, as well as the observed price fluctuation intensity up to this time τ. Further model requirements include the existence of closed-form formulas or at least fast valuation schemes for European options, a small number of parameters providing sufficient model flexibility, and reliable and fast parameter identification from historical data. The goal of this paper is to present and analyze simple models satisfying these requirements.

Our starting point is the non-compliance event $N \in \mathcal{F}_T$ which we describe as the event where a hypothetic positive-valued random variable Γ_T exceeds the boundary 1

$$N = \{\Gamma_T \geq 1\}.$$

If one denotes by E_T the total pollution within the period $[0, T]$ which must be balanced against the total number $\gamma \in (0, \infty)$ of credits issued by the regulator, then the event of non-compliance is given by

$$N = \{E_T \geq \gamma\}$$

which suggests that Γ_T should be viewed as the normalized total emission E_T/γ. However, in our modeling, we merely describe the non-compliance event. Strictly speaking, so any random variable Γ_T with

$$\{\Gamma_T \geq 1\} = \{E_T/\gamma \geq 1\}$$

would do as well. On this account, we do not claim that Γ_T represents the total normalized emission E_T/γ.

So we will consider the martingale

$$A_t = \pi \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\mathcal{F}_t), \quad t \in [0, T]$$

where the random variable Γ_T is chosen from a suitable parameterized family of random variables

$$\{\Gamma_T^\theta \,:\, \theta \in \Theta\}. \tag{2}$$

For reasons of model tractability, we suppose that the filtered probability space supports a process $(W_t)_{t \in [0,T]}$ of Brownian motion with respect to the spot martingale measure $\mathbb{Q}$, and we investigate families (2) which give allowance prices

$$A_t^\theta = \pi \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T^\theta \geq 1\}} \,|\mathcal{F}_t), \quad t \in [0, T]$$

with a Markovian stochastic evolution of the form

$$dA_t^\theta = v^\theta(t, A_t^\theta) dW_t$$

where the diffusion term v^θ captures the basic properties of historical allowance prices. In particular, we will match exactly the observed initial allowance price and the initial instantaneous price fluctuation intensity.

To simplify the notation, we consider the normalized futures price process

$$a_t := \frac{1}{\pi} A_t = \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\, \mathcal{F}_t) \qquad t \in [0, T].$$

and we describe it under special assumptions on Γ_T. Our goal is to identify classes of martingales $\{a_t\}_{t \in [0,T]}$ taking values in the interval $(0, 1)$, and satisfying

$$\mathbb{P}\{\lim_{t \nearrow T} a_t = 1\} = 1 - \mathbb{P}\{\lim_{t \nearrow T} a_t = 0\} \in (0, 1). \tag{3}$$

We first identify a parametric family of such martingales by working backward from a simple model for the random variable Γ_T which may be interpreted as a proxy for the final cumulative level of emissions.

2.1 Basic Model

Throughout the paper we use the notation $N(\mu, \sigma^2)$ for tee normal distribution with mean μ and variance σ^2, and write Φ to denote the distribution function of the standard normal distribution.

Proposition 1. *Suppose that*

$$\Gamma_T = \Gamma_0 e^{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds}, \qquad \Gamma_0 \in (0, \infty) \tag{4}$$

for some continuous and square-integrable deterministic function $(0,T) \ni t \hookrightarrow \sigma_t$*. Then the martingale*

$$a_t = \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\, \mathcal{F}_t) \quad t \in [0,T] \tag{5}$$

is given by

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \tag{6}$$

and it solves the stochastic differential equation

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{z_t} dW_t \tag{7}$$

where the positive-valued function $(0,T) \ni t \hookrightarrow z_t$ *is given by*

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \qquad t \in (0,T). \tag{8}$$

Proof. A direct calculation shows

$$\begin{aligned}
a_t &= \mathbb{E}^{\mathbb{Q}}(1_{\{\Gamma_T \geq 1\}} \,|\, \mathcal{F}_t) = \mathbb{Q}\{\Gamma_T \geq 1 \,|\, \mathcal{F}_t\} \\
&= \mathbb{Q}\left\{\Gamma_t e^{\int_t^T \sigma_s dW_s - \frac{1}{2}\int_t^T \sigma_s^2 ds} \geq 1 \,|\, \mathcal{F}_t\right\} \\
&= \Phi\left(\frac{\ln \Gamma_t - \frac{1}{2}\int_t^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) + \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}} \frac{\sqrt{\int_0^T \sigma_s^2 ds}}{\sqrt{\int_t^T \sigma_s^2 ds}} + \frac{\int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right),
\end{aligned}$$

and taking into account the initial condition

$$a_0 = \Phi\left(\frac{\ln(\Gamma_0) - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}}\right),$$

we obtain the desired expression (6). In order to show (7), we start with (6)

$$a_t = \Phi(X_t), \qquad t \in [0, T],$$

where

$$X_t = \frac{x_0 + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}} \qquad \text{for } t \in [0, T], \quad X_0 = x_0 = \Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} \tag{9}$$

with deterministic $a_0 \in (0, 1)$. Computing its Itô differential, we get

$$\begin{aligned} dX_t &= (\int_t^T \sigma_s^2 ds)^{-\frac{1}{2}} \sigma_t dW_t + (x_0 + \int_0^t \sigma_s dW_s)(-\frac{1}{2})(\int_t^T \sigma_s^2 ds)^{-\frac{3}{2}}(-1)\sigma_t^2 dt \\ &= (\int_t^T \sigma_s^2 ds)^{-\frac{1}{2}} \sigma_t dW_t + \frac{1}{2} X_t (\int_t^T \sigma_s^2 ds)^{-1} \sigma_t^2 dt \\ d[X]_t &= (\int_t^T \sigma_s^2 ds)^{-1} \sigma_t^2 \end{aligned}$$

and we derive the differential of the normalized allowance prices as

$$\begin{aligned} da_t &= d\Phi(X_t) = \Phi'(X_t) dX_t + \frac{1}{2}\Phi''(X_t) d[X]_t \\ &= \Phi'(X_t) dX_t + \frac{1}{2}(-X_t)\Phi'(X_t) d[X]_t \\ &= \Phi'(X_t)(\int_t^T \sigma_s^2 ds)^{-\frac{1}{2}} \sigma_t dW_t + \Phi'(X_t)\frac{1}{2} X_t (\int_t^T \sigma_s^2 ds)^{-1} \sigma_t^2 dt \\ &\quad + \frac{1}{2}\Phi'(X_t)(-X_t)(\int_t^T \sigma_s^2 ds)^{-1} \sigma_t^2 \\ &= \Phi'(X_t)(\int_t^T \sigma_s^2 ds)^{-\frac{1}{2}} \sigma_t dW_t \\ &= \Phi'(\Phi^{-1}(a_t))\sqrt{z_t} dW_t \end{aligned}$$

with positive-valued function z_t defined by (8). □

The stochastic differential equation (7) can be interpreted in the following way. Because of the factor $\sqrt{z_t}$ in front of dW_t, a_t can be viewed as the time-changed version of a martingale $\{Y_t\}_{t \geq 0}$ indexed by $[0, \infty)$, taking values in $(0, 1)$ and satisfying

$$\mathbb{P}\{\lim_{t \nearrow \infty} Y_t = 1\} = 1 - \mathbb{P}\{\lim_{t \nearrow \infty} Y_t = 0\} \in (0, 1) \tag{10}$$

This can be considered as a special case of a general program where the martingale $\{a_t\}_{t \in [0,T]}$ satisfying (3) is constructed in two steps: first determine a martingale $\{Y_t\}_{t \in [0,\infty)}$ satisfying (10), and then the search for a time change bringing the half-axis $[0, \infty)$ onto the bounded interval $[0, T)$. With this in mind, it appears natural to consider the solutions of the stochastic differential equation

$$dY_t = \Phi'(\Phi^{-1}(Y_t)) dW_t, \quad Y_0 \in (0, 1), \qquad t \geq 0 \tag{11}$$

and more generally solutions of the stochastic differential equation

$$dY_t = \Theta(Y_t)dW_t, \quad Y_0 \in (0,1), \qquad\qquad t \geq 0, \tag{12}$$

where Θ is a nonnegative continuous function on $[0,1]$ satisfying $\Theta(0) = \Theta(1) = 0$. We can then use Feller's classification (see for example [10] or [11]) to check that such a diffusion is conservative, does not reach the boundaries 0 and 1 in finite time, and satisfies (10). This is indeed the case if $v(0+) = v(1-) = \infty$ where $v(x)$ is defined by

$$v(x) = 2\int_{0.5}^{x}(x-y)\frac{dy}{\Theta(y)}, \qquad x \in (0,1)$$

Straightforward computations show that the solution of the stochastic differential equation (12) does indeed satisfy these conditions. In other words, the solution $(Y_t)_{t\in[0,\infty)}$ does not hit 0 and 1 in finite time with probability one.

2.1.1 More Examples

For the sake of completeness, we conclude this section with the discussion of a couple of examples of martingales in $[0,1]$ which satisfy (3). This subsection can be skipped in a first reading. Indeed, because of the intuitive meaning attached to the definition (5), we will use the martingale model for $(a_t)_{t\in[0,T]}$ introduced in Proposition 1 throughout the paper.

We first note that the stochastic differential equation

$$dX_t = \sqrt{2}dW_t + X_t dt$$

with initial condition $X_0 = x_0$ has the solution

$$X_t = e^t\big(x_0 + \sqrt{2}\int_0^t e^{-s}dW_s\big)$$

and that:

$$\lim_{t\to\infty} X_t = +\infty \qquad \text{on the set } \{\int_0^\infty e^{-s}dW_s > -x_0\}$$
$$\lim_{t\to\infty} X_t = -\infty \qquad \text{on the set } \{\int_0^\infty e^{-s}dW_s < -x_0\}$$

Moreover, a direct application of Ito's Lemma shows that the process $\{Y_t\}_{t\in[0,\infty)}$ defined by

$$Y_t = \Phi(X_t), \qquad t \in [0,\infty)$$

is a martingale. Moreover, it clearly satisfies (10). Notice that

$$dY_t = \sqrt{2}\Phi'(X_t)\,dW_t = \sqrt{2}\Phi'(\Phi^{-1}(Y_t))\,dW_t$$

which shows that Y_t is a solution of the stochastic differential equation (12) with $\Theta(x) = \sqrt{2}\Phi'(\Phi^{-1}(x))$ for all $x \in (0,1)$.

Similar explicit examples can be constructed from the analysis of [2] which we learned from Mike Terhanchi (who extended the argument of [2] to Lévy processes in [21]). If we now set

$$X_t = e^{-W_t+ct}\left(X_0 - \int_0^t e^{W_s-cs}(a\,ds + dB_s)\right), \qquad X_0 \in \mathbb{R}$$

for some constants $c > 0$ and $a \in \mathbb{R}$, and where $\{W_t\}_{t\in[0,\infty)}$ and $\{B_t\}_{t\in[0,\infty)}$ are independent Wiener processes, then $\{X_t\}_{t\in[0,\infty)}$ satisfies

$$dX_t = \left[\left(c+\frac{1}{2}\right)X_t - a\right]dt\, - X_t dW_t - dB_t, \qquad t \in [0,\infty)$$

and

$$\lim_{t\to\infty} X_t = -\infty \qquad \text{on the set } \{\int_0^\infty e^{W_s-cs}(a\,ds+dB_s) > X_0\},$$
$$\lim_{t\to\infty} X_t = +\infty \qquad \text{on the set } \{\int_0^\infty e^{W_s-cs}(a\,ds+dB_s) < X_0\}.$$

Now if we define the function G by $G(x) = \int_{-\infty}^x g(y)dy$ for all $x \in \mathbb{R}$ where the function g is

$$g(y) = C\frac{e^{2a\tan^{-1}y}}{(1+y^2)^{c+1/2}}, \qquad\qquad y \in \mathbb{R},$$

with the constant $C > 0$ chosen so that $\int_{-\infty}^{+\infty} g(y)dy = 1$, then it is easy to check that

$$\frac{1}{2}g'(y)(1+y^2) + g(y)[(c+\frac{1}{2})y - a] = 0$$

which in turn implies that $Y_t = G(X_t)$ is a martingale. Clearly, this martingale satisfies the limits (10). Moreover, a simple application of Itô's formula shows that $\{Y_t\}_t$ is a solution of the stochastic differential equation (12) with $\Theta(y) = g(G^{-1}(y))\sqrt{1+G^{-1}(y)^2}$.

3 Model Parametrization and Calibration

We now show how to calibrate the model introduced in the previous section. Note that the choice of the function $(0,T) \ni t \hookrightarrow \sigma_s$ affects only the time-change part of the model. In other words, our model is universal in the sense that it is completely determined up to a deterministic time change. Moreover, Proposition 1 shows that when modeling the random variable Γ_T by (4), we must assume that the function $(0,T) \ni t \hookrightarrow \sigma_s$ is not constant. Indeed, a constant volatility

$$\sigma_s \equiv \bar\sigma \in (0,\infty) \quad \text{for all } \ s \in [0,T]$$

would give, independently on the choice of $\bar\sigma$, the same process

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{T} +\ W_t}{\sqrt{T-t}}\right) \tag{13}$$

with dynamics

$$da_t = \Phi'(\Phi^{-1}(a_t))\frac{1}{\sqrt{T-t}}dW_t. \tag{14}$$

Thus, with a constant and deterministic $\bar{\sigma}$ it is impossible to match both, the recent allowance price and the recently observed (instantaneous) fluctuation intensity. This suggests that we introduce two degrees of freedom to (14) resulting in the model

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{\beta(T-t)^{-\alpha}}dW_t \tag{15}$$

with additional parameters $\alpha \in \mathbb{R}$, $\beta \in (0, \infty)$. At this stage, it is not clear which α and β correspond to a given choice of $(\sigma_s)_{s\in[0,T]}$. In what follows, we solve this problem determining the function family

$$(\sigma_s(\alpha, \beta))_{s\in(0,T)}, \qquad \alpha \geq 1, \beta > 0, \tag{16}$$

such that with the choice $(\sigma_s = \sigma_s(\alpha, \beta))_{s\in(0,T)}$ the process $(a_t)_{t\in[0,T]}$ defined by (5) actually satisfies (15). Furthermore, we show how to calibrate the parameterized family (16) to historical data.

As seen from (4), the function $(0,T) \ni s \hookrightarrow \sigma_s$ enters the dynamics of $(a_t)_{t\in[0,T]}$ in (7) indirectly, through the time-change function $(0,T) \ni t \hookrightarrow z_t$ defined in (8). The correspondence between the functions σ and z is identified in the following

Lemma 1. *a) Given square-integrable continuous and positive function $(0,T) \ni s \hookrightarrow \sigma_s$, the function $(0,T) \ni t \hookrightarrow z_t$ defined by*

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \qquad t \in (0,T), \tag{17}$$

satisfies

$$(z_t)_{t\in(0,T)} \quad \text{is positive, continuous and} \quad \lim_{t\nearrow T}\int_0^t z_u du = +\infty. \tag{18}$$

b) Conversely, if the function $(0,T) \ni t \hookrightarrow z_t$ satisfies (18) then there exists a square integrable, positive-valued and continuous function $(0,T) \ni s \hookrightarrow \sigma_s$ which satisfies (17).

Proof. a) Write (17) as $z_t\varphi_t = \sigma_t^2$ for $t \in (0,T)$ where

$$\varphi_t = \int_t^T \sigma_u^2 du \quad \text{for all } t \in [0,T]$$

which implies that $\dot{\varphi}_t = -\sigma_t^2$ for $t \in (0,T)$ and ensures to conclude that φ satisfies the following differential equation:

$$z_t\varphi_t = -\dot{\varphi}_t \qquad t \in (0,T).$$

The solution to this equation is given by

$$\varphi_t = \varphi_0 e^{-\int_0^t z_u du} \qquad t \in [0, T),$$

and form the terminal condition $\varphi_T = \int_T^T \sigma_u^2 du = 0$, we conclude that

$$\lim_{t \uparrow T} \int_0^t z_u du = +\infty.$$

b) Let us now suppose that $(z_t)_{t\in(0,T)}$ satisfies (18), and let us define the positive and continuous function $(\varphi_t)_{t\in[0,T)}$ by

$$\varphi_t = e^{-\int_0^t z_u du} \qquad t \in [0, T).. \tag{19}$$

It satisfies

$$\dot{\varphi}_t = -z_t \varphi_t, \qquad t \in (0, T), \tag{20}$$

hence, we have

$$-z_t = \frac{\dot{\varphi}_t}{\varphi_t} \qquad t \in (0, T). \tag{21}$$

Now, observe that the divergence of the integral ensures the existence of the limit

$$\varphi_T = \lim_{t \uparrow T} e^{-\int_0^t z_u du} = 0.$$

which yields

$$\varphi_t = -\int_t^T \dot{\varphi}_u du \qquad t \in (0, T). \tag{22}$$

With this representation, (21) is given as

$$-z_t = \frac{\dot{\varphi}_t}{\varphi_t} = -\frac{\dot{\varphi}_t}{\int_t^T \dot{\varphi}_u du}.$$

Thus, with definition

$$\sigma_t^2 = -\dot{\varphi}_t \qquad t \in (0, T) \tag{23}$$

we obtain the representation (17). Moreover, this function (23) is positive and continuous due to (19) and (20) and also integrable because of $1 = \varphi(0) = -\int_0^T \dot{\varphi}_u du$ which follows from (19) and (22). Consequently, the function defined by $\sigma_t := \sqrt{\sigma_t^2}$ for $t \in (0, T)$ is a square integrable, continuous, and positive. Furthermore it represents $(z_t)_{t\in(0,T)}$ by (17), as required. □

We return to the expression (4) for Γ_T, using now the targeted family (16) to determine the stochastic differential equation (15). In light of the previous lemma, the function

$$(z_t(\alpha, \beta) = \beta(T - t)^{-\alpha})_{t\in(0,T)},$$

must satisfy (18), implying the following restrictions on parameters $\alpha, \beta \in \mathbb{R}$

$$\beta > 0 \quad \text{and} \quad \alpha \geq 1. \tag{24}$$

In what follows, we use the fact that β is a multiplicative parameter in the sense that

$$z_t(\alpha, \beta) = \beta z_t(\alpha, 1), \quad t \in (0, T), \;\; \beta > 0, \;\; \alpha \geq 1. \tag{25}$$

The function $(\sigma_t(\alpha, \beta))_{t \in (0,T)}$ associated to $(z_t(\alpha, \beta))_{t \in [0,T]}$ via previous lemma is given by:

$$\begin{aligned} \sigma_t(\alpha, \beta)^2 &= z_t(\alpha, \beta) e^{-\int_0^t z_u(\alpha,\beta) du} && (26) \\ &= \begin{cases} \beta (T-t)^{-\alpha} e^{\beta \frac{T^{-\alpha+1} - (T-t)^{-\alpha+1}}{-\alpha+1}} & \text{for } \beta > 0, \alpha > 1 \\ \beta (T-t)^{\beta - 1} T^{-\beta} & \text{for } \beta > 0, \alpha = 1 \end{cases} && (27) \end{aligned}$$

for all $t \in (0, T)$.

3.1 Historical Calibration

Consider historical observation of the futures prices $(A_t)_{t \in [0,T]}$, recorded at times

$$t_1 < t_2 <, \ldots, < t_{n+1}.$$

resulting in a data set

$$(a_{t_i}(\omega) = \frac{1}{\pi} A_{t_i}(\omega))_{i=1}^{n+1}. \tag{28}$$

Assuming dynamics of the form

$$da_t = \Phi'(\Phi^{-1}(a_t)) \sqrt{z_t(\alpha, \beta)} dW_t$$

we use (25) to conclude that:

$$\frac{da_t}{\Phi'(\Phi^{-1}(a_t))} = \sqrt{z_t(\alpha, 1)} \sqrt{\beta} dW_t \qquad t \in [0, T], \tag{29}$$

showing that we should be able to determine the parameters $\beta > 0$ and $\alpha \geq 1$ which best explain the data (28). Let us fix $\alpha \geq 1$ momentarily to derive closed-form maximum likelihood estimate for $\beta > 0$. For the purpose of statistical estimation from historical data, we need to work with the objective measure $\mathbb{P}$ which can be recovered from the spot martingale measure $\mathbb{Q}$ via its Radon-Nikodym density

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{\int_0^T H_t dW_t - \frac{1}{2} \int_0^T H_t^2 dt}.$$

For the sake of simplicity, we shall assume that the market price of risk process $(H_t)_{t \in [0,T]}$ is constant and deterministic with value

$$H_t \equiv h, \qquad t \in [0, T]$$

for some fixed $h \in \mathbb{R}$. Girsanov's theorem implies that the process $(W_t - ht)_{t \in [0,T]}$ is a Brownian motion with respect to the objective measure $\mathbb{P}$, and under the measure $\mathbb{P}$, a_t satisfies a stochastic differential equation with the same volatility and a new drift. The

Euler scheme for this new stochastic differential equation justifies the introduction for $i = 1, \dots, n$, of the quantities

$$\begin{aligned} Y_i &= \frac{a_{t_{i+1}} - a_{t_i}}{\Phi'(\Phi^{-1}(a_{t_i}))} \\ \Delta_i &= t_{i+1} - t_i \\ z_i(\alpha) &= z_{t_i}(\alpha, 1) \end{aligned} \tag{30}$$

which can be computed from the observations (28), and for which the following assumptions can be made

$$\left.\begin{array}{c} \text{Under } \mathbb{P}\text{, the random variables } (Y_i)_{i=1}^n \text{ are} \\ \text{independent and normally distributed with} \\ Y_i \sim N(\sqrt{z_i(\alpha)}\sqrt{\beta}h\Delta_i, z_i(\alpha)\beta\Delta_i) \\ \text{for all } i = 1, \dots, n. \end{array}\right\} \tag{31}$$

Given assumption (31) for fixed $\alpha \geq 1$, the distribution of $(Y_i)_{i=1}^n$ depends only on the unknown parameters $h \in \mathbb{R}$ and $\beta > 0$. Evaluated on realization $(y_i)_{i=1}^n \in \mathbb{R}^n$, the log-likelihood density (computed with respect to Lebesgue measure on $\mathbb{R}^n$) is

$$L_{y_1,\dots,y_n}(h, \beta, \alpha) = \sum_{i=1}^n \left(-\frac{(y_i - \sqrt{z_i(\alpha)}\sqrt{\beta}h\Delta_i)^2}{2\Delta_i\beta z_i(\alpha)} - \ln(\sqrt{2\pi\Delta_i\beta z_i(\alpha)}) \right) \tag{32}$$

for all $h \in \mathbb{R}$, $\beta > 0$, and $\alpha \geq 1$. For fixed $\alpha \geq 1$, the maximum-likelihood estimation of the parameters $h \in \mathbb{R}$ and $\beta \in (0, \infty)$ can be computed explicitly.

Lemma 2. *For fixed $\alpha \geq 1$ the maximum of $(h, \beta) \to L_{y_1,\dots,y_n}(h, \beta, \alpha)$ on $(h, \beta) \in \mathbb{R} \times (0, \infty)$ is attained at the values $(h^*, \beta^*) = (h^*(\alpha), \beta^*(\alpha))$ given in terms of*

$$x^* = \left(\sum_{i=1}^n \frac{y_i}{\sqrt{z_i(\alpha)}} \right) \left(\sum_{i=1}^n \Delta_i \right)^{-1}$$

by

$$\beta^* = \frac{1}{n} \sum_{i=1}^n \frac{(y_i - \Delta_i\sqrt{z_i(\alpha)}x^*)^2}{\Delta_i z_i(\alpha)}, \tag{33}$$

$$h^* = \frac{1}{\sqrt{\beta^*}} x^* \tag{34}$$

Proof. For fixed $\alpha \geq 1$, fix $\beta \in (0, \infty)$ momentarily in order to compute $h(\beta, \alpha)$ as the maximizer of $h \mapsto L_{y_1,\dots,y_n}(h, \beta, \alpha)$. It is obtained by solving

$$\frac{\partial}{\partial h} L_{y_1,\dots,y_n}(h, \beta, \alpha) = \sum_{i=1}^n \frac{2(y_i - \sqrt{z_i(\alpha)}\sqrt{\beta}h\Delta_i)}{2\Delta_i\beta z_i(\alpha)} \sqrt{z_i(\alpha)}\sqrt{\beta}\Delta_i = 0.$$

which is equivalent to

$$\sum_{i=1}^n \frac{y_i}{\sqrt{z_i(\alpha)}\sqrt{\beta}} = \sum_{i=1}^n h\Delta_i \quad \Rightarrow \quad h(\beta) = \frac{1}{\sqrt{\beta}} \left(\sum_{i=1}^n \frac{y_i}{\sqrt{z_i(\alpha)}} \right) \left(\sum_{i=1}^n \Delta_i \right)^{-1}.$$

Now plug the expression

$$h(\beta, \alpha) = \frac{1}{\sqrt{\beta}} x^* \tag{35}$$

into the formula (32) of the likelihood density and find $\beta^*(\alpha)$ as the maximizer of $\beta \mapsto L_{y_1,\dots,y_n}(h(\beta,\alpha), \beta, \alpha)$ on $\beta \in (0, \infty)$ by solving

$$\frac{\partial}{\partial \beta} L_{y_1,\dots,y_n}(h(\beta,\alpha),\beta,\alpha) = \sum_{i=1}^{n} \left(\frac{(y_i - \sqrt{z_i(\alpha)}\Delta_i x^*)^2}{2\Delta_i \beta^2 z_i(\alpha)} - \frac{1}{2\beta} \right) = 0$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{(y_i - \sqrt{z_i(\alpha)}\Delta_i x^*)^2}{\Delta_i z_i(\alpha)} = n\beta^*$$

and gives (33). Finally, β^* must be plugged into (35) to obtain (34). □

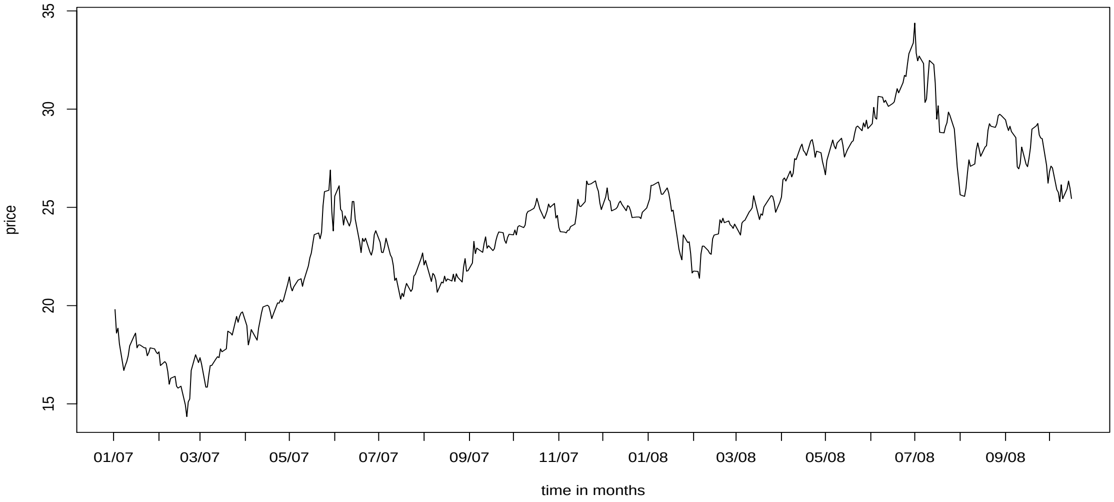

Figure 1: Future prices on EUA with maturity Dec. 2012

To illustrate this result, we set $\alpha = 1$ and computed the above estimates for the futures prices reproduced on Figure 1, of the European Union Allowance for the second phase of the European Emission Trading Scheme.

Normalizing these historical futures prices as in (28) with $\pi = 100$, we extracted a realization $(y_i)_{i=1}^n \in \mathbb{R}^n$ of the series $(Y_i)_{i=1}^n$ defined in (30). The estimations (34) produced the values

$$h^* = h^*(1) = 0.4656, \qquad \beta^* = \beta^*(1) = 0.4377.$$

To verify the validity of our procedure, we determine

$$w_i = \frac{y_i - \sqrt{z_i(1)}\sqrt{\beta^*} h^* \Delta_i}{\sqrt{z_i(1)\beta^*\Delta_i}}, \qquad i = 1, \dots, n,$$

Standard statistical analysis of these *residuals* confirm the validity of our approach.

Finally, consider the identification of $\alpha \geq 1$. Although there is no closed-form estimate for this parameter, the maximum of the likelihood function can be determined numerically. After plugging in $(h^*(\alpha), \beta^*(\alpha))$ from Lemma 2 into the likelihood function, we need to find the maximizer α^* of $L_{y_1,\dots,y_n}(h^*(\alpha), \beta^*(\alpha), \alpha)$ over $\alpha \geq 1$. Figure 2 gives the plot of this likelihood as a function of α.

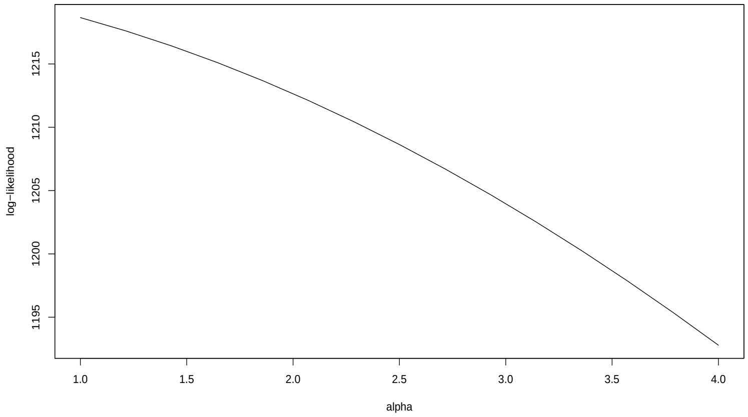

Figure 2: The maximum at $\alpha^* = 1$ of $\alpha \mapsto L_{y_1,\dots,y_n}(h^*(\alpha), \beta^*(\alpha), \alpha)$.

Obviously, $\alpha = 1$ is a local maximum. Apparently, the allowance price data from Figure 1 give no support to the assumption that $\alpha > 1$. On this account, we suppose for the reminder of this work that $\alpha = 1$. We work with the one-parameter family

$$\sigma_t(\beta)^2 = \beta(T-t)^{\beta-1}T^{\beta}, \qquad t \in (0,T), \qquad \beta > 0 \tag{36}$$

to describe the risk-neutral allowance price time-evolution, and address the problem of option pricing under this standing assumption.

3.2 Option Pricing

Now, we turn our attention to the valuation of European call options written on allowance futures price $(A_t)_{t\in[0,T]}$. The payoff of a European call with strike price $K \geq 0$ and maturity $\tau \in [0,T]$ is given by $(A_\tau - K)^+$ at maturity $\tau \in [0,T]$. Under the assumption that the savings account $\{B_t\}_{t\in[0,T]}$ is given by $B_t = e^{\int_0^t r_s B_s ds}$ for $t \in [0,T]$ for with deterministic short rate $\{r_s\}_{s\in[0,T]}$, this price can be computed in the model proposed in this paper. Indeed, an arbitrage-free price at time $t = 0$ is

$$C_0 = \frac{B_\tau}{B_T}\mathbb{E}^{\mathbb{Q}}\{(A_\tau - K)^+\}. \tag{37}$$

Now we use the explicit form of the allowance futures price $A_\tau = \pi\Phi(X_\tau)$ with

$$X_\tau = \Phi^{-1}(A_0/\pi)\sqrt{\frac{\int_0^T \sigma_s^2 ds}{\int_\tau^T \sigma_s^2 ds}} + \frac{\int_0^\tau \sigma_s dW_s}{\sqrt{\int_\tau^T \sigma_s^2 ds}} \tag{38}$$

to conclude that the normal distribution $N(\mu_\tau, \nu_\tau)$ of the random variable X_τ determines the price (37) as

$$C_0 = e^{-\int_0^\tau r_s ds} \int_{\mathbb{R}} (\pi\Phi(x) - K)^+ N(\mu_\tau, \nu_\tau^2)(dx). \tag{39}$$

Let us compute the mean $\mu_\tau = \mathbb{E}\{X_\tau\}$ and the variance $\nu_\tau^2 = \mathrm{Var}\{X_\tau\}$. According to the standing assumption $\alpha = 1$, we have

$$\begin{aligned} \sigma_t^2 &= \beta(T-t)^{\beta-1}, \\ z_t(1,\beta) &= \beta(T-t)^{-1}, \qquad \beta > 0, \quad t \in (0,T). \end{aligned}$$

(We did not use T^β from (36) in the first equation since a constant factor cancels out in the fraction (17).) From this, we obtain

$$\int_\tau^T \sigma_s^2 ds = \frac{\sigma_\tau^2}{z_\tau} = (T-\tau)^\beta, \tag{40}$$

which implies that for all $t \in [0,T]$

$$\sqrt{\frac{\int_0^T \sigma_s^2 ds}{\int_\tau^T \sigma_s^2 ds}} = \sqrt{\left(\frac{T}{T-\tau}\right)^\beta}, \qquad \text{and} \qquad \frac{\int_0^\tau \sigma_s^2 ds}{\int_\tau^T \sigma_s^2 ds} = \left(\frac{T}{T-\tau}\right)^\beta - 1. \tag{41}$$

With this, we derive the mean and the variance of X_τ from (38):

$$\begin{aligned} \mu_\tau &= \Phi^{-1}(A_0/\pi)\sqrt{\left(\frac{T}{T-\tau}\right)^\beta}, \\ \nu_\tau &= \left(\frac{T}{T-\tau}\right)^\beta - 1. \end{aligned}$$

In general, at any time $t \in [0,\tau]$ prior to maturity, the price of the call is obtained similarly:

Proposition 2. *In a one-compliance period model, with parameters $\alpha = 1$, $\beta > 0$, the price of European call with strike price $K \geq 0$ written on allowance futures price at time $\tau \in [0,T]$ is given at time $t \in [0,\tau]$ by*

$$\begin{aligned} C_t &= e^{-\int_t^\tau r_s ds} E^{\mathbb{Q}}((A_\tau - K)^+ \,|\, \mathcal{F}_t), \\ &= \int_{\mathbb{R}} (\pi\Phi(x) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx) \end{aligned}$$

with

$$\mu_{t,\tau} = \Phi^{-1}(A_t/\pi)\sqrt{\left(\frac{T-t}{T-\tau}\right)^\beta}, \qquad \textit{and} \qquad \nu_{t,\tau} = \left(\frac{T-t}{T-\tau}\right)^\beta - 1 \tag{42}$$

This result is obtained by a straightforward calculation using

$$X_\tau = \Phi^{-1}(A_t/\pi)\sqrt{\frac{\int_t^T \sigma_s^2 ds}{\int_\tau^T \sigma_s^2 ds}} + \frac{\int_t^\tau \sigma_s dW_s}{\sqrt{\int_\tau^T \sigma_s^2 ds}} \quad 0 \le t \le \tau \le T,$$

instead of (38).

Let us illustrate the role of the parameter β on option prices. In the following example, we assume that the penalty is $\pi = 100$ and we suppose that at the initial time $t = 0$ four years prior to the compliance date $T = 4$ the price of a futures contract, written on allowance price at T is $A_0 = 25$. For constant and deterministic continuously compounded interest rate $r = 0.05$ we consider European calls written on futures price A_τ at an intermediate date $\tau \in [0, T]$ with strike price of $K = 25$, whose price is to be calculated by (39). We now illustrate the dependence of the option price upon the maturity of the option and and parameter β. Comparing three cases $\beta = 0.5$, $\beta = 0.8$ and $\beta = 1.1$, Figure 3.2 shows that the call price is increasing in β. Less surprisingly, the dependence on τ shows that longer-maturity calls (with the same strike) are more valuable than their short-maturity counterparts.

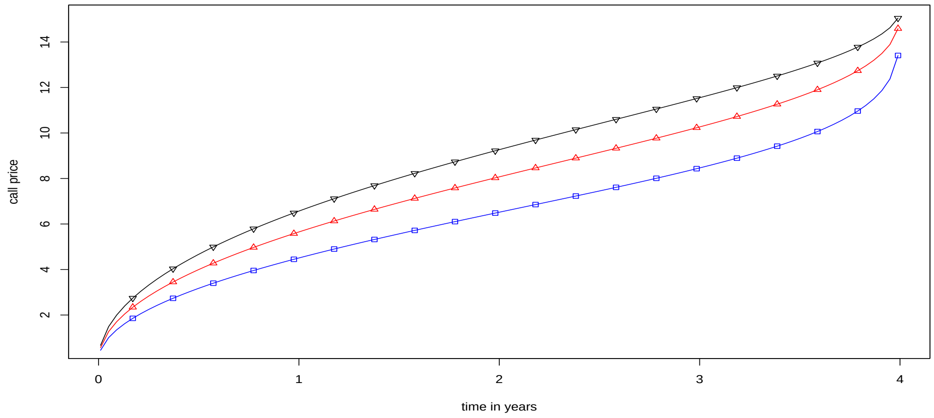

Figure 3: Plots of the prices $C_0(\tau)$ at time $t = 0$ as functions of option maturity τ. The graphs marked by □, △, and ▽ stand for $\beta = 0.5$, $\beta = 0.8$, and $\beta = 1.1$ respectively. The values of the other parameters are given in the text.

Let us stress the fact that, although there are no closed-form formulas for call prices, their numerical evaluations can be performed very efficiently.

4 Multi-Compliance Periods Markets

So far, we focused on a generic cap and trade scheme modeled after the first phase of the EU ETS, namely limited to one compliance period and without banking in the sense that unused allowances become worthless at the end of the period. This is a strong simplification since as already mentioned above, real-world markets are operating in a multi-period framework. Furthermore, subsequent periods are connected by regulations, which are market design-specific. In what follows, we consider an abstract but generic model of such a market and focus on most natural rules for the period interconnection.

Presently, there are three regulatory mechanisms connecting successive compliance periods in a cap-and-trade scheme. Their rules go under the names of *borrowing*, *lending* and *withdrawal*.

- Borrowing allows for the transfer of a (limited) number of allowances from the next period into the present one;
- Banking allows for the transfer of a (limited) number of (unused) allowances from the present period into the next;
- Withdrawal penalizes firms which fail to comply in two ways: by penalty payment for each unit of pollutant which is not covered by credits and by withdrawal of the missing allowances from their allocation for the next period.

From the nature of the existing markets and the designs touted for possible implementation, it seems that policy makers tend to admit unlimited banking and forbid borrowing. Furthermore, the withdrawal rule is most likely to be included. Banking and withdrawal seem to be reasonable rules to reach an emission target within a fixed number of periods, because each success (resp. failure) in the previous period results in stronger (resp. weaker) abatement in the subsequent periods.

4.1 Market Model

For the remainder of this section, we consider a two-period market model without borrowing, with unlimited banking and with withdrawal. On this account, we introduce two periods $[0, T]$ and $[T, T']$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T']})$ equipped with a distinct measure $\mathbb{Q} \sim \mathbb{P}$ which we view as the spot martingale measure. Further, we introduce processes $(A_t)_{t\in[0,T]}$, $(A'{}_t)_{t\in[0,T]}$ for the futures contracts with maturities at compliance dates T, T' written on allowance prices from the first and the second period respectively. In order to exclude arbitrage, we suppose that the prices $(A_t)_{t\in[0,T]}$ and $(A'_t)_{t\in[0,T']}$ are martingales with respect to the spot martingale measure $\mathbb{Q}$. Non-compliance in the first and in the second periods occur on events $N \in \mathcal{F}_T$ and $N' \in \mathcal{F}_{T'}$ respectively. As before, we assume that the savings account $(B_t)_{t\in[0,T']}$ is given by

$$B_t = e^{\int_0^t r_s ds}, \qquad t \in [0, T'] \tag{43}$$

for some deterministic short rate $(r_s)_{s\in[0,T']}$. The results of the previous section imply that in the case $\Omega\setminus N$ of the first-period compliance the allowance price drops

$$A_T 1_{\Omega\setminus N} = \kappa A'_T 1_{\Omega\setminus N}, \tag{44}$$

where $\kappa\in(0,\infty)$ stands for discount factor describing the interest rate effect

$$\kappa = B_T B_{T'}^{-1} = e^{-\int_T^{T'} r_s ds}.$$

The relation (44) is justified by considering spot prices. The random variable $\kappa A'_T$ is nothing but the spot price at time T of the second-period allowance. Because futures and spot price agree at maturity, A_T must be the spot price of the first period allowance at T. In the case of compliance in the first period, the unused allowances can be banked, hence we have the equality in (44).

In the case of non-compliance at the end of the first period, the withdrawal regulation implies that

$$A_T 1_N = \kappa A'_T 1_N + \pi 1_N. \tag{45}$$

Namely, the non-compliance in one pollutant unit at time T costs a penalty π in addition to one allowance from the next period which must be withdrawn at the spot price $\kappa A'_T$.

Combining the results (44) and (45) we find out that the difference is

$$A_t - \kappa A'_t = \mathbb{E}^{\mathbb{Q}}(A_T - \kappa A'_T \,|\, \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(\pi 1_N \,|\, \mathcal{F}_t) \qquad t\in[0,T]$$

and must be modeled as $\{0,\pi\}$-valued martingale. We suggest to use the same methodology as in one period model

$$A_t - \kappa A'_t = \pi\Phi(X^1_t) \qquad t\in[0,T], \tag{46}$$

where the Gaussian process $(X^1_t)_{t\in[0,T]}$ is introduced as previously in (9), with $(\sigma_s)_{s\in[0,T]}$ in parameterized form (27) and driven by a process $(W^1_t,\mathcal{F}_t)_{t\in[0,T']}$of Brownian motion.

To model the second-period allowance futures price, a continuation of the cap-and-trade system must be specified. If there is no agreement on long-term regulatory framework (as it is the case for the most of the existing emission markets), the process $(A'_t)_{t\in[0,T]}$ should be specified exogenously. The simplest choice would be a Geometric Brownian motion with constant volatility. Another idea to handle the uncertain continuation is to suppose that the cap and trade system will be terminated after the second period. In this case,

$$A'_t = \mathbb{E}^{\mathbb{Q}}(\pi 1_{N'} \,|\, \mathcal{F}_t) \qquad t\in[0,T']$$

can also be modeled as in the one-period model

$$A'_t = \pi\Phi(X^2_t) \qquad t\in[0,T']. \tag{47}$$

Again, $(X^2_t)_{t\in[0,T]}$ is introduced as in (9), with a process $\{\sigma^2_s\}_{s\in[0,T]}$ chosen in parameterized form (27) and driven by another Brownian motion $(W^2_t,\mathcal{F}_t)_{t\in[0,T']}$.

4.2 Option Pricing

As an application of our two-period model, we consider pricing of European Calls. Consider European Call option with strike price $K \geq 0$ and maturity $\tau \in [0, T]$ written on futures price of allowance from the first period. This contract yields a payoff

$$C_\tau = (A_\tau - K)^+ \qquad \text{at time } \tau \in [0, T].$$

Under the assumptions of the previous section, we start with the computation of the price C_0

$$C_0 = e^{-\int_0^\tau r_s ds} \mathbb{E}^{\mathbb{Q}}((A_\tau - K)^+)$$

of this option at time $t = 0$. Using the decomposition

$$(A_\tau - K)^+ = (A_\tau - \kappa A'_\tau + \kappa A'_\tau - K)^+,$$

we utilize our modeling of $\{0, \pi\}$-valued martingales (46) and (47) to express the terminal payoff as

$$(A_\tau - K)^+ = (\pi \Phi(X^1_\tau) + \kappa \pi \Phi(X^2_\tau) - K)^+$$

with expectation

$$\begin{aligned} C_0 &= e^{-\int_0^\tau r_s ds} \mathbb{E}^{\mathbb{Q}}((A_\tau - K)^+) \\ &= e^{-\int_0^\tau r_s ds} \mathbb{E}^{\mathbb{Q}}((\pi \Phi(X^1_\tau) + \kappa \pi \Phi(X^2_\tau) - K)^+) \\ &= e^{-\int_0^\tau r_s ds} \int_{\mathbb{R}^2} (\pi \Phi(x_1) + \kappa \pi \Phi(x_2) - K)^+ N(\mu_\tau, \nu_\tau)(dx_1, dx_2) \end{aligned} \tag{48}$$

where $N(\mu_\tau, \nu_\tau) = F_{X^1_\tau, X^2_\tau}$ stands for joint normal distribution of X^1_τ and X^2_τ.

Let us derive the mean μ_τ and the covariance matrix ν_τ under the standing assumption $\alpha_1 = \alpha_2 = 1$, for $\beta_1 > 0$, $\beta_2 > 0$. We have

$$\begin{aligned} X^1_\tau &= \Phi^{-1}(\frac{A_0 - \kappa A'_0}{\pi}) \sqrt{(\frac{T}{T - \tau})^{\beta_1}} + \beta_1^{\frac{1}{2}} \frac{\int_0^\tau (T - u)^{\frac{\beta_1 - 1}{2}} W^1_u du}{(T - \tau)^{\frac{\beta_1}{2}}} \\ X^2_\tau &= \Phi^{-1}(\frac{\kappa A'_0}{\pi}) \sqrt{(\frac{T'}{T' - \tau})^{\beta_2}} + \beta_2^{\frac{1}{2}} \frac{\int_0^\tau (T' - u)^{\frac{\beta_2 - 1}{2}} W^2_u du}{(T' - \tau)^{\frac{\beta_2}{2}}}. \end{aligned}$$

Denoting by ρ the correlation of the two Brownian motions $(W^1_t)_{t \in [0, T']}$ and $(W^2_t)_{t \in [0, T']}$

$$[W^1, W^2]dt = \rho dt, \qquad \rho \in [-1, 1],$$

we can apply the same argumentation to obtain the means

$$\begin{aligned} \mu^1_\tau &= E(X^1_\tau) = \Phi^{-1}(\frac{A_0 - \kappa A'_0}{\pi}) \sqrt{(\frac{T}{T - \tau})^{\beta_1}}, \\ \mu^2_\tau &= E(X^2_\tau) = \Phi^{-1}(\frac{\kappa A'_0}{\pi}) \sqrt{(\frac{T'}{T' - \tau})^{\beta_2}}, \end{aligned}$$

the variances

$$\begin{aligned}\nu_\tau^{1,1} &= \mathrm{Var}(X_\tau^1) = \left(\frac{T}{T-\tau}\right)^{\beta_1} - 1,\\ \nu_\tau^{2,2} &= \mathrm{Var}(X_\tau^2) = \left(\frac{T'}{T'-\tau}\right)^{\beta_1} - 1,\end{aligned}$$

and the covariance as

$$\nu_\tau^{1,2} = \nu_\tau^{2,1} = \mathrm{Cov}(X_\tau^1, X_\tau^2) \quad = \quad \beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} \frac{\int_0^\tau (T-u)^{\frac{\beta_1-1}{2}}(T'-u)^{\frac{\beta_2-1}{2}}\rho du}{(T-\tau)^{\frac{\beta_1}{2}}(T'-\tau)^{\frac{\beta_2}{2}}}.$$

At times $t \in [0,\tau]$ prior to maturity, the price C_t of the call is obtained similarly:

Proposition 3. *In a two-compliance periods model as above, with parameters $\beta_1, \beta_2 > 0$ and $\rho \in (-1,1)$, the price of the European Call with strike price $K \geq 0$ and maturity $\tau \in [0,T]$ written on first-period allowance futures price is given at time $t \in [0,\tau]$ by*

$$C_t = e^{-\int_t^\tau r_s ds} \int_{\mathbb{R}^2} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx_1, dx_2) \tag{49}$$

with mean $\mu_{t,\tau}$

$$\mu_{t,\tau}^1 \quad = \quad \Phi^{-1}\left(\frac{A_t - \kappa A'_t}{\pi}\right)\sqrt{\left(\frac{T-t}{T-\tau}\right)^{\beta_1}} \tag{50}$$

$$\mu_{t,\tau}^2 \quad = \quad \Phi^{-1}\left(\frac{\kappa A'_t}{\pi}\right)\sqrt{\left(\frac{T'-t}{T'-\tau}\right)^{\beta_2}} \tag{51}$$

and covariance matrix $\nu_{t,\tau}$

$$\nu_{t,\tau}^{1,1} \quad = \quad \mathrm{Var}(X_\tau^1) = \left(\frac{T-t}{T-\tau}\right)^{\beta_1} - 1 \tag{52}$$

$$\nu_{t,\tau}^{2,2} \quad = \quad \mathrm{Var}(X_\tau^2) = \left(\frac{T'-t}{T'-\tau}\right)^{\beta_1} - 1 \tag{53}$$

$$\nu_{t,\tau}^{1,2} = \nu_{t,\tau}^{2,1} \quad = \quad \beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} \frac{\int_t^\tau (T-u)^{\frac{\beta_1-1}{2}}(T'-u)^{\frac{\beta_2-1}{2}}\rho du}{(T-\tau)^{\frac{\beta_1}{2}}(T'-\tau)^{\frac{\beta_2}{2}}}. \tag{54}$$

If we take a closer look at the computations involved in the valuation of the call price

$$C_t = C_t(\tau, T, T', A_0, A'_0, K, r, \beta_1, \beta_2, \rho)$$

given by the formulas (49) – (54), we see that because of

$$C_t(\tau, T, T', A_0, A'_0, K, r, \beta_1, \beta_2, \rho) = C_0(\tau - t, T - t, T' - t, A_0, A'_0, K, r, \beta_1, \beta_2, \rho)$$

for all $t \in [0,\tau]$, it suffices to consider the case $t = 0$. The numerical evaluation of two dimensional integral is easily performed by using a decomposition of the two-dimensional normal distribution. To ease the notation, let us skip t, τ to write $\mu^i = \mu^i_{t,\tau}$, $\nu^{i,j} = \nu^{i,j}_{t,\tau}$ for $i, j = 1, 2$. It holds

$$N(\mu, \nu)(dx_1, dx_2) = N(\mu^{1,c}(x_2), \nu^{1,1,c})(dx_1) N(\mu^2, \nu^{2,2})(dx_2) \tag{55}$$

where the conditional mean and the conditional variance are given by

$$\begin{aligned} \mu^{1,c}(x_2) &= \mu^1 + \frac{\nu^{2,1}}{\nu^{2,2}}(x_2 - \mu^2) \\ \nu^{1,1,c} &= \nu^{1,1} - \frac{(\nu^{2,1})^2}{\nu^{2,2}} \end{aligned}$$

With factorization (55), the inner integral is calculated explicitly in the following cases

$$\begin{aligned} &\int_{\mathbb{R}} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)^+ N(\mu^{1,c}(x_2), \nu^{1,1,c})(dx_1) \\ &= \left\{ \begin{array}{cll} 0 & \text{if} & K - \pi\kappa\Phi(x_2) \geq \pi \\ \pi\Phi(\sqrt{\frac{\mu^{1,c}(x_2)}{1+\nu^{1,1,c}}}) + \pi\kappa\Phi(x_2) - K & \text{if} & K - \pi\kappa\Phi(x_2) \leq 0 \end{array} \right. . \end{aligned}$$

That is, the numerical valuation is required only in the case $0 < K - \pi\kappa\Phi(x_2) < \pi$ where

$$\int_{\Phi^{-1}(K/\pi - \kappa\Phi(x_2))}^{\infty} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K) N(\mu^{1,c}(x_2), \nu^{1,1,c})(dx_1)$$

needs to be calculated.

Having obtained the inner integral, the numerical evaluation of the outer integral is straightforward. Since the density of the normal distribution decays sufficiently fast, we do not expect neither numerical difficulties nor long computation times. In fact, we did not encounter any problem implementing this formula.

For the sake of completeness, we illustrate the dependence of the call price on β_1 and maturity of the call. To make the results comparable with the one-period example given above, we chose the following parameters: four years to the first-period compliance date $T = 4$, eight years to the second-period compliance date $T_2 = 8$, initial first-period allowance futures price is $A_0 = 25$, initial second-period allowance futures price is $A'_0 = 15$, strike price of the European call is $K = 25$, interest rate $r = 0.05$, and $\beta_2 = 0.2$. Figure 4.2 depicts the dependence of the call price on the value of β_1 for the first period and of the call maturity τ.

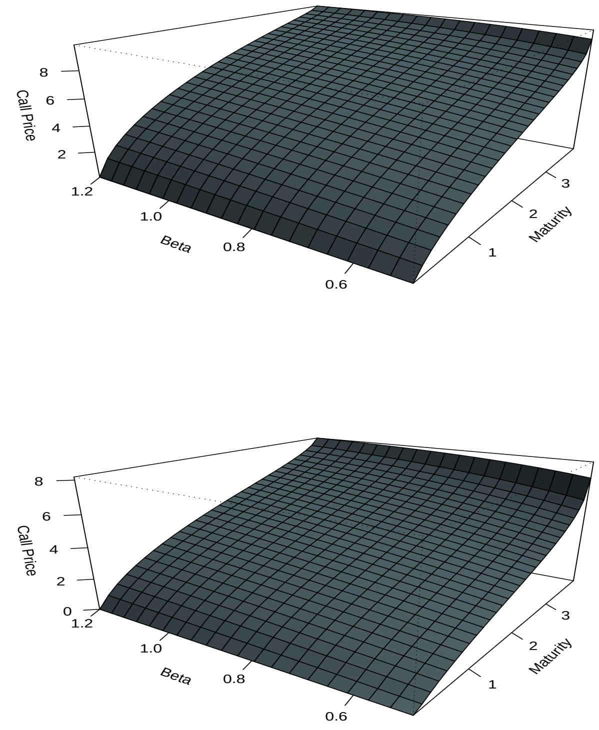

Figure 4: Surface plots of the initial call price $(\tau, \beta_1) \mapsto C_0(\tau, \beta_1)$ as function of maturity τ and β_1 for correlation $\rho = 0.8$ (top) and $\rho = -0.8$ (bottom). The values of the other parameters are given in the text.

5 Conclusion

Mandatory emission markets are being established throughout the world. In the most mature market, the European Emission Trading Scheme, beyond physical allowances, a large volume of allowance futures is traded. Furthermore, European options written on these futures are introduced and traded although no theoretical foundation for their pricing is available yet.

The goal of this work is to fill this gap. In our analysis, we gradually move from one-period market model to a more realistic situation of two-period markets (covering the present EU ETS regulations) and show that martingales finishing at two-valued random variables can be considered as basic building blocks which form the risk-neutral futures price dynamics. We suggest a model for two-valued martingales, flexible in terms of time- and space changing volatility and capable to match the observed historical or implied volatility of the underlying future. From hedging perspective, this issue could be one of the most desirable model properties. Other practical aspects like ease of calibration and simple option valuation schemes are also fulfilled in our approach. We show how parameters can be estimated from historical price observation and suggest efficient option valuation schemes. Although option price formulas are not available in a closed form, a simple and fast numerical integration can be applied.

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6 Appendix: An implementation in R

Let us elaborate on the implementation aspects in the valuation of European Calls on Emission allowances.

The function for the one-period formula

$$\begin{aligned} C_t &= e^{-\int_t^\tau r_s ds} E^{\mathbb{Q}}((A_\tau - K)^+ \mid \mathcal{F}_t) \\ &= \int_{\mathbb{R}} (\pi \Phi(x) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx) \end{aligned}$$

where

$$\begin{aligned}\mu_{t,\tau} &= \Phi^{-1}(A_t/\pi)\sqrt{\left(\frac{T-t}{T-\tau}\right)^{\beta}} \\ \nu_{t,\tau} &= \left(\frac{T-t}{T-\tau}\right)^{\beta} - 1.\end{aligned}$$

is realized by

```
Call1<-function(ta, Tmat , A, K, r, beta1)
```

with parameters

- `ta` corresponds to $\tau - t$, time to options maturity
- `Tmat` stands for $T - t$, time to the final compliance
- `penalty, A, K, r, beta1` correspond to the model parameters π, A_t, K, r, β

The implementation code is

```
Call1<-function(ta, Tmat , A, K, r, beta1)
{  mu<-qnorm(A/penalty)*(Tmat/(Tmat-ta))^(beta1/2)
   nu<-(Tmat/(Tmat-ta))^beta1-1
 f<-function(x)
 {(penalty*pnorm(x)-K)*dnorm(mean=mu, sd=sqrt(nu), x) }
 return(exp(-r*ta)*integrate(f, qnorm(K/penalty), Inf)$value)
}
```

The function for the two-period formula

$$C_t = e^{-\int_t^\tau r_s ds} \int_{\mathbb{R}^2} (\pi\Phi(x_1) + \kappa\pi\Phi(x_2) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx_1, dx_2)$$

with mean $\mu_{t,\tau}$

$$\begin{aligned} \mu^1_{t,\tau} &= \Phi^{-1}(\frac{A_t - \kappa A'_t}{\pi})\sqrt{(\frac{T-t}{T-\tau})^{\beta_1}} \\ \mu^2_{t,\tau} &= \Phi^{-1}(\frac{\kappa A'_t}{\pi})\sqrt{(\frac{T'-t}{T'-\tau})^{\beta_2}} \end{aligned}$$

and covariance matrix $\nu_{t,\tau}$

$$\begin{aligned} \nu^{1,1}_{t,\tau} &= \mathrm{Var}(X^1_\tau) = \left(\frac{T-t}{T-\tau}\right)^{\beta_1} - 1 \\ \nu^{2,2}_{t,\tau} &= \mathrm{Var}(X^2_\tau) = \left(\frac{T'-t}{T'-\tau}\right)^{\beta_1} - 1 \\ \nu^{1,2}_{t,\tau} = \nu^{2,1}_{t,\tau} &= \beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} \frac{\int_t^\tau (T-u)^{\frac{\beta_1-1}{2}}(T'-u)^{\frac{\beta_2-1}{2}}\rho du}{(T-\tau)^{\frac{\beta_1}{2}}(T'-\tau)^{\frac{\beta_2}{2}}}. \end{aligned}$$

is realized by

```
Call1<-function(ta, Tmat1, Tmat2 , A1,A2, K, r, beta1, beta2, rho)
```

with parameters

- `ta` corresponds to $\tau - t$, time to options maturity
- `Tmat1` stands for $T_1 - t$, time to the first period compliance
- `Tmat2` stands for $T_2 - t$, time to the second period compliance
- `A1` stands for A_t, first-period allowance futures price
- `A2` stands for A'_t, second-period allowance futures price
- `beta1, beta2` stands for β and β' respectively
- `rho` denotes the correlation ρ
- `penalty, K, r,` correspond to the model parameters π, K, r

The implementation code is

```
Call2<-function(ta, Tmat1, Tmat2 , A1,A2, K, r, beta1, beta2, rho)
{
kapp<-exp(-r*(Tmat2-Tmat1))
mu1<-qnorm((A1-kapp*A2)/penalty)*(Tmat1/(Tmat1-ta))^(beta1/2)
mu2<-qnorm((kapp*A2)/penalty)*(Tmat2/(Tmat2-ta))^(beta2/2)
nu11<- (Tmat1/(Tmat1-ta))^(beta1) -1
nu22<- (Tmat2/(Tmat2-ta))^(beta2) -1
g<-function(u)
{ (Tmat1-u)^((beta1-1)/2)*(Tmat2-u)^((beta2-1)/2)
}
nu12<-sqrt(beta1*beta2/((Tmat1-ta)^(beta1)*
(Tmat2-ta)^(beta2)))*integrate(g, 0, ta)$value*rho
nu1c<-nu11-(nu12)^2/nu22

GG<-function(x)
{
      mu1c<-mu1+(x-mu2)*(nu12/nu22)
      Kc<-K-kapp*penalty*pnorm(x)
     if (Kc>=penalty)
         result<-0
     if (Kc<=0)
        result<-penalty*pnorm(mu1c/sqrt(1+nu1c))-Kc
     if ((0<Kc)&(Kc<penalty))
     {
      f<-function(x)
      {(penalty*pnorm(x)-Kc)*dnorm(mean=mu1c, sd=sqrt(nu1c), x)}
     result<-integrate(f, qnorm(Kc/penalty), Inf)$value
     }
     return(result*dnorm(mean=mu2, sd=sqrt(nu22),x))
}

GGG<-Vectorize(GG)
return(exp(-r*ta)*integrate(GGG, -Inf, Inf)$value)
}
```